International Journal of Theoretical Physics, Vol. 32, No. 4, 1993

Generalized Clifford Algebras and Hyperspin Manifolds

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Received February 25, 1992

We consider a special extension of Clifford algebras and show that these generalized Clifford algebras are naturally equipped with a metric defined by a fundamental form of degree n which is $SL(n, \mathbb{C}) \otimes SL(n, \mathbb{C})$ invariant. Using the embedding of the quaternions in the generalized Clifford algebras, in the Hermitian limit, we obtain an algebraic description of the inclusion of the Minkowski space into the hyperspin manifold.

The aim of this paper is to establish a connection between generalized Clifford algebras (GCA) and hyperspin manifolds (HM). The associative GCA have been studied and classified by Morinaga and Nono (1952). Yamazaki (1964), Popovici and Ghéorghe (1966*a,b*), and Morris (1967, 1968). The HM have been defined by Finkelstein (1986) and Finkelstein *et al.* (1986) to reconcile the ideas of Kaluza–Klein theory with the notion of spin manifold. They built a space supplied with an $n-ic$ metric which reduces, in a special limit, to the Minkowski space-time.

Using results of Cartan (1898), who established that quaternions are included in GCA, we obtain, after providing GCA with an appropriate metric, the algebraic approach of the space-time description mentioned above.

This paper is organized as follows: in Section 1 we briefly review some properties of the GCA and the Cartan inclusion. Section 2 is devoted to the construction of an n-scalar product of elements of the GCA. This defines an n-metric which is shown to be $SL(n, \mathbb{C}) \otimes SL(n, \mathbb{C})$ invariant. In Section 3,

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0020-7748/93/0400-0503$07.00/0

we first recall the arguments leading to the concept of hyperspin manifold. We next obtain its algebraic description using the generalized Clifford numbers in the Hermitian limit.

1. PRELIMINARIES

The GCA $\mathscr{C}_p^{(n)}$ is generated by a set of p canonical generators $e_1, \ldots, e_p$ fulfilling

$$e_i e_j = \omega^{\mathrm{sg}(j-i)} e_j e_i, \qquad e_i^n = 1, \qquad i, j = 1, \ldots, p \tag{1.1}$$

where $\omega = \exp(2i\pi/n)$ is an nth primitive root of unity and $\mathrm{sg}(x)$ is the usual sign function. $\mathscr{C}_p^{(n)}$ is an extension of the usual $n=2$ Clifford algebra. The case of one generator as well as the associated trigonometry has been investigated in more detail by Fleury *et al.* (1991). The $p=2$ case was considered at the end of the last century to extend quaternions ($n=2$) to nonions ($n=3$) by Sylvester and Clifford (see Cartan, 1909, pp. 206 and 217 and references therein) and also to n^2-ions by Sylvester (1884) and Cartan (1898). Those algebras ($p=2$) were rediscovered by Weyl (1932) and led to quantum mechanics in finite discrete space; then Schwinger (1960) proved that such operators define a complete basis of unitary operators. When $p>2$ (the case $n=2$ excepted), these algebras have to be considered as $\mathbb{C}$-algebras. Because $\sqrt{\omega} = \zeta$ is also a complex number, $\mathscr{C}_p^{(n)}$ could have been equivalently defined with the condition $e_i^n = -1$. The special case when $\sqrt{\omega}$ does not belong to the field on which $\mathscr{C}_p^{(n)}$ is built has been investigated by Morris (1967, 1968) and Thomas (1974).

The $\mathbb{R}$-algebra $\mathscr{C}_2^{(2)}$ with the definition $e_i^2 = -1$ ($i = 1, 2$) is, as mentioned, the quaternions, while the $\mathbb{C}$-algebra case gives the biquaternions. Let us mention, for instance, that the Pauli numbers $x = x_0 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$ (σ_i are the Pauli matrices) cannot define an $\mathbb{R}$-algebra because $\sigma_1\sigma_2 = i\sigma_3$.

In the following, we will consider the $\mathbb{C}$-algebra $\mathscr{C}_2^{(n)}$:

$$\mathscr{C}_2^{(n)} = \left\{ x = \sum_{a,b=0}^{n-1} x_{ab} e_1^a e_2^b;\ x_{ab} \in \mathbb{C} \right\}$$

It has been shown by Cartan (1898) that for integers $m<n$, $\mathscr{C}_2^{(m)}$ is included in $\mathscr{C}_2^{(n)}$. It has also been proved by Morinaga and Nono (1952), Popovici and Ghéorghe (1966*b*), and Morris (1967, 1968) that $\mathscr{C}_2^{(n)}$ is isomorphic to the set of $n \times n$ complex matrices. A proof can be given as follows: Consider the n^2 quantities

$$h_{ab} = \frac{1}{n} \sum_{k=0}^{n-1} \omega^{-ak} e_2^k e_1^{b-a}; \qquad 0 \le a,\ b \le n-1 \tag{1.2}$$

They constitute an equivalent basis of $\mathscr{C}_n^{(2)}$, as can be easily checked. Their multiplication law is

$$h_{ab}h_{cd}=h_{ad}\delta_{bc} \tag{1.3}$$

and the mapping

$$h_{ab}\xrightarrow{f} H_{ab} \tag{1.4a}$$

where H_{ab} is the $n\times n$ matrix with elements

$$(H_{ab})_{ij}=\delta_{ai}\delta_{jb} \tag{1.4b}$$

is an isomorphism. This provides a matricial representation for e_1 and e_2:

$$E_1=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & \ddots & \ddots & & \vdots\\ \vdots & & \ddots & \ddots & \vdots\\ & & & \ddots & 1\\ 1 & \cdots & \cdots & \cdots & 0\end{pmatrix},\qquad E_2=\begin{pmatrix} \omega & & & 0\\ & \omega^2 & & \\ & & \ddots & \\ 0 & & & 1\end{pmatrix}$$

These matrices play a central role in the literature on generalized Clifford algebras.

To achieve our goal, we will, in Cartan's way, define the $\mathscr{C}_2^{(m|n)}$ space as the one generated by h_{ab} with the condition $0\le a,\ b\le m-1$. The two canonical generators of $\mathscr{C}_2^{(m|n)}$ are obtained with the inverse of the transformation (1.2):

$$\begin{aligned} f_1&=\sum_{k=0}^{m-1} h_{k+1,k} \qquad (\text{with } h_{m,m-1}\equiv h_{0,m-1})\\ f_2&=\sum_{k=0}^{m-1} \theta^k h_{kk}\end{aligned} \tag{1.5}$$

where we used $\theta=\exp(2i\pi/m)$. One can check that

$$\begin{aligned} f_1\cdot f_2&=\theta f_2\cdot f_1\\ f_1^m=f_2^m&=\sum_{k=1}^{m-1} h_{kk}=P^{m|n}\end{aligned} \tag{1.6}$$

The operator $P^{m|n}$ projects $\mathscr{C}_2^{(n)}$ on $\mathscr{C}_2^{(m|n)}$. It has the matrix representation

$$\begin{pmatrix} 1 & & & & & 0\\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 0\end{pmatrix}\begin{matrix}\Big\}m\\ \\ \Big\}n-m\end{matrix}$$

$\mathscr{C}_2^{(m|n)}$ defines a m^2-dimensional $\mathbb{C}$-algebra with $P^{m|n}$ its neutral element for the multiplication. But it is not a subalgebra of $\mathscr{C}_2^{(n)}$. Instead, consider the (m^2+1)-dimensional $\mathbb{C}$-algebra:

$$\tilde{\mathscr{C}}_2^{(m|n)} = \mathscr{C}_2^{(m|n)} \oplus \mathbb{C}(1 - P^{m|n}) \tag{1.7}$$

with elements

$$\tilde{x} = x + \lambda \sum_{k=m+1}^{n-1} h_{kk} = x + \lambda(1 - P^{m|n})$$

where x is an element of $\mathscr{C}_2^{(m|n)}$ and λ an ordinary complex number. We have the relations

$$\begin{aligned} f_1 \cdot f_2 &= \theta f_2 \cdot f_1 \\ (1 - P^{m|n})^2 &= 1 - P^{m|n} \\ (1 - P^{m|n}) \cdot f_i &= f_i(1 - P^{m|n}) = 0 \end{aligned} \tag{1.8}$$

Obviously, the case $m = 2$ provides a natural inclusion of biquaternions in $\mathscr{C}_2^{(n)}$. In his definition of n^2-ions, Cartan used the basis h_{ab} instead of $e_1^a e_2^b$. We conclude this section by noting that similar arguments lead to the inclusion of $\mathscr{C}_2^{(m|n)} \oplus \mathscr{C}_2^{(n-m|n)}$ in $\mathscr{C}_2^{(n)}$.

2. CONSTRUCTION OF A METRIC ON GCA. ITS SYMMETRY GROUP

The definition of a metric on Clifford or more generally Cayley–Dickson algebras leads naturally to a quadratic metric (see, for instance, Wene, 1984). This is no longer true for GCA, for which the metric is a homogeneous form of degree n [see (1.1)]. The $p = 1$ case has already been handled by Fleury *et al.* (1991). We will follow the same procedure for the $\mathscr{C}_2^{(n)}$ space. For x, y elements of $\mathscr{C}_2^{(n)}$, their product is

$$z = \left(\sum_{a,b=0}^{n-1} x_{ab} e_1^a e_2^b \right) \left(\sum_{c,d=0}^{n-1} y_{cd} e_1^c e_2^d \right) = \sum_{A,B=0}^{n-1} z_{AB} e_1^A e_2^B \tag{2.1a}$$

with

$$z_{AB} = \sum_{a,b=0}^{n-1} \tilde{\Delta}_{Aa,Bb}(x) y_{ab}\,; \qquad \tilde{\Delta}_{Aa,Bb}(x) = x_{A-a,B-b}\, \omega^{-a(B-b)} \tag{2.1b}$$

Had we taken $e_1^n = e_2^n = -1$ in the definition of $\mathscr{C}_2^{(n)}$, we would have obtained an overall minus sign in the previous equation.

Just as for the case $p=1$, the problem of the inverse leads us to solve the equation

$$x \cdot y = \tilde{\Delta}(x)y = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Proposition 2.1. We have

$$\det \tilde{\Delta}(x) = [\det \Delta(x)]^n$$

where $\Delta(x)$ is the $n \times n$ matrix with coefficients

$$\Delta_{Aa}(x) = \sum_{i=0}^{n-1} x_{A-a,i}\,\omega^{-ai}; \qquad A, a = 0, \ldots, n-1$$

Proof. Looking at the $n^2 \times n^2$ matrix $\tilde{\Delta}(x)$ of (2.1b), we see that it has the structure of a direct product:

$$\tilde{A}(x) = \sum_i M_i \otimes N_i$$

where M and N are now $n \times n$ matrices. The first couple of indices (Aa) is related to the matrix M and the second (Bb) to the matrix N. Noticing that $\tilde{\Delta}$ depends only on the difference $B-b$, using the matrix E_1 of Section 1, one has in fact

$$\tilde{\Delta}(x) = \sum_{i=0}^{n-1} X_i \otimes E_1^i$$

with

$$(X_i)_{Aa} = x_{A-a,i}\,\omega^{-ai}$$

If we diagonalize E_1, we obtain E_2 and $\tilde{\Delta}(x)$ becomes

$$\tilde{\Delta}'(x) = \sum_{i=0}^{n-1} X_i \otimes E_2^i$$

which has a diagonal block structure:

$$\tilde{\Delta}'(x) = \begin{pmatrix} \Delta_1(x) & & & \\ & \Delta_2(x) & & 0 \\ & & \ddots & \\ & 0 & & \Delta_n(x) \end{pmatrix}$$

with

$$(\Delta_j(x))_{Aa} = \sum_{i=0}^{n-1} x_{A-a,i}\omega^{-i(a+j)}$$

The $n \times n$ Δ_j matrices satisfy

$$(\Delta_{j+1}(x))_{ab} = (\Delta_j(x))_{a+1,b+1}$$

and hence

$$\det \Delta_1(x) = \det \Delta_2(x) = \cdots = \det \Delta_n(x) = \det \Delta(x)$$

And finally

$$\det \tilde{\Delta}(x) = \det \tilde{\Delta}'(x) = [\det \Delta(x)]^n \quad \blacksquare$$

Note that the use of the isomorphism (1.4) leads to $X = \Delta(x)$ as a matrix representation of x, and to Proposition 2.1; but it will be convenient not to utilize the matrix representation, as we will see later. We can use $\Delta(x)$ to provide $\mathscr{C}_2^{(n)}$ with a pseudonorm:

$$\|x\|^{2n} = \det[\Delta(x)\Delta^+(x)] \tag{2.2}$$

In Section 3 this will reduce to

$$\|x\|^n = \det \Delta(x)$$

For a quaternion q, element of $\mathscr{C}_2^{(n)}$, one gets, for instance

$$\|q\|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$$

Due to the property of $\tilde{\Delta}$

$$\tilde{\Delta}(x_1 x_2) = \tilde{\Delta}(x_1)\tilde{\Delta}(x_2) \tag{2.3}$$

it turns out that $\mathscr{C}_2^{(n)}$ is a composition algebra, but not an integer algebra. Moreover, if x is not invertible, $\det \tilde{\Delta}(x) = 0$ and so $\tilde{\Delta}$ is degenerate. In accordance with the Hurwitz theorem, quaternions and complex numbers are the only integer and nondegenerate algebras. From the homogeneous form $\det \Delta$, one can build an n-linear symmetric form allowing the calculation of the scalar product of n different vectors (see, for instance, Kwasniewski, 1985).

Consider the direct sum $\mathscr{C}_2^{(m)} \oplus \mathscr{C}_2^{(n-m|n)}$ with element $x_1 \oplus x_2$. What is the consequence on the metric of such a splitting? Call $y = y_1 \oplus y_2$ its inverse element. Because

$$x_1 x_2 = x_2 x_1 = 0$$

for (x_1, x_2) element of $\mathscr{C}_2^{(m)} \times \mathscr{C}_2^{(n-m|n)}$, one has

$$xy = x_1 y_1 \oplus x_2 y_2 = 1 = P^{(m|n)} \oplus (1 - P^{(m|n)})$$

The consequence on the metric is

$$\Delta(x_1 \oplus x_2) = \begin{pmatrix} \Delta_m(x_1) & 0 \\ 0 & \Delta_{n-m}(x_2) \end{pmatrix} \tag{2.4}$$

where

$$(\Delta_m(x_1))_{Aa} = \sum_{i=0}^{n-1} x_{1_{A-a,i}} \theta^{-ai} \tag{2.4a}$$

$$(\Delta_{n-m}(x_2))_{Aa} = \sum_{i=0}^{n-m-1} x_{2_{A-a,i}} \rho^{-ai} \tag{2.4b}$$

are respectively $m \times m$ and $(n-m) \times (n-m)$ matrices, $\theta = \exp(2i\pi/m)$, and $\rho = \exp[2i\pi/(n-m)]$. As a special case, consider $x_2 = \lambda(1 - P^{(m|n)})$ [that is, the only component of x_2 different from zero is $(x_2)_{00}$]. Then,

$$\Delta(x_1, \lambda) = \lambda^{n-m} \Delta_m(x_1) \tag{2.5}$$

and taking $\lambda = 1$ and $m = 2$, one has for the Pauli numbers

$$\Delta(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2$$

which is just the Minkowski metric. With the isomorphism (1.4a) one gets the spinorial construction of space-time of Cartan (1938).

The determinantal property of Proposition 2.1 is a consequence of the algebraic structure of $\mathscr{C}_2^{(n)}$ and can be understood in the context of the smash product (Long, 1976)

$$\mathscr{C}_2^{(n)} = \mathscr{C}_1^{(n)} \# \mathscr{C}_1^{(n)}$$

which is a twisted tensorial product reproducing (1.1).

The next point of this section is the study of the symmetry group of the n-metric det Δ

Proposition 2.2. $SL(n, \mathbb{C}) \otimes SL(n, \mathbb{C})$ is the symmetry group of det Δ.

Proof. It is obvious that $\mathscr{C}_2^{(n)}$ can be considered as a sum of $\mathscr{C}_1^{(n)}$-type algebras. Notice that this sum is not direct. Take one of these algebras, $\mathscr{C}_{1(a)}^{(n)}$, generated by $e_{(a)}$. Consider the element of $\mathscr{C}_{1(a)}^{(n)}$:

$$R_a = \exp\left\{ \sum_{p=1}^{n-1} \varphi_p e_{(a)}^p \right\} \tag{2.6}$$

and write the complex number φ_p as

$$\varphi_p = x_p + \zeta^p y_p$$

Then R_a factorizes into "real" and "imaginary" parts:

$$(R)_{ar} = \exp\left\{\sum_{p=1}^{n-1} x_p e_{(a)}^p\right\} = \sum_{p=0}^{n-1} \mathrm{mush}_p(x) e_{(a)}^p$$

$$(R)_{ai} = \exp\left\{\sum_{p=1}^{n-1} y_p \zeta^p e_{(a)}^p\right\} = \sum_{p=0}^{n-1} \mathrm{mus}_p(y) \zeta^p e_{(a)}^p$$

where we use the mus and mush functions introduced by Fleury *et al.* (1991)

$$\mathrm{mus}_i(x) = \sum_{\alpha, p=0}^{\infty} \frac{(-1)^\alpha}{p!} \sum_{\substack{i_1 \ldots i_p = 1 \\ i_1 + \cdots + i_p = n\alpha + i}}^{n-1} x_{i_1} \ldots x_{i_p}$$

$$\mathrm{mush}_i(x) = \sum_{\alpha, p=0}^{\infty} \frac{1}{p!} \sum_{\substack{i_1 \ldots i_p = 1 \\ i_1 + \cdots + i_p = n\alpha + i}}^{n-1} x_{i_1} \ldots x_{i_p}$$

which are an extension of the usual circular and hyperbolic functions and can be connected to the one-variable trigonometric and hyperbolic functions of order n (Edérlyi, 1955).

For a Cartesian element of $\mathscr{C}_{1(a)}^{(n)}$:

$$x_{(a)} = \sum_{i=1}^{n-1} x_i e_{(a)}^i$$

the associated determinant reduces to the one of the circulant matrix:

$$\Delta_{(a)}(x) = \begin{pmatrix} x_0 & x_{n-1} & \cdots & x_1 \\ x_1 & x_0 & \cdots & x_2 \\ \vdots & & & \vdots \\ x_{n-1} & \cdots & \cdots & x_0 \end{pmatrix}$$

Now, with the relations

$$\det \Delta_{(a)} \; (\zeta^i \, \mathrm{mus}_i(x)) = 1$$

$$\det \Delta_{(a)} \; (\mathrm{mush}_i(y)) = 1$$

one gets, just like R_a, that $(R)_{ai}$ and $(R)_{ar}$ are unimodular numbers. Summing on all the values of a in the exponential formulation of R_a, one gets that $R = R_{a=1} R_{a=2} \ldots R_{a=a_f}$ is also unimodular. With the Hausdorff–Campbell formula, one writes

$$R = \exp \sum_{\substack{i,j=0 \\ (i,j)\neq(0,0)}}^{n-1} \varphi_{ij}\, e_1^i e^j{}_2$$

as a unimodular number. Moreover, any unimodular number can be written in this form since it is not singular, it has an exponential representation [nonsingular complex matrices have a defined logarithm and unimodular numbers can be represented by the use of (1.4) by nonsingular matrices]. Then, if

$$\exp\left\{\sum_{i,j=0}^{n-1} \varphi_{ij} e_1^i e_2^j\right\} = \rho \exp \sum_{\substack{i,j=0 \\ (i,j)\neq(0,0)}}^{n-1} \varphi_{ij} e_1^i e_2^j$$

using $\det \Delta(x) = 1$, one gets that $\rho = 1$ and hence

$$U = \left\{ R = \exp \sum_{\substack{i,j=0 \\ (i,j)\neq(0,0)}}^{n-1} \varphi_{ij} e_1^i e_2^j;\ \varphi_{ij} \in \mathbb{C} \right\}$$

in the set of unimodular numbers.

We can prove, moreover, that the most general transformation on x leaving $\det \Delta(x)$ invariant is given by two unimodular numbers acting respectively on the left and right sides:

$$R_{R,L} = \exp\left\{ \sum_{\substack{i,j=0 \\ (i,j)\neq(0,0)}}^{n-1} \varphi_{L,R_{ij}}\, e_1^i e_2^j \right\}$$

and $x' = R_L x R_R$ is such that $\det \Delta(x') = \det \Delta(x)$. Indeed, the symmetry group of $\det \Delta(x)$ can be identified using h_{ab} and the isomorphism f, (1.4a). From the canonical basis H_{ab}, one builds the generators T_a of SU(n) represented by $f^{-1}(T_a) = t_a$. Because φ_{ij} is complex, $R_{R,L}$ is an element of $SL(n, \mathbb{C})$ and $\det \Delta(x)$ is invariant under the $SL_L(n, \mathbb{C}) \otimes SL_R(n, \mathbb{C})$ transformations. ■

As previously mentioned, Proposition 2.1 could have been more straightforwardly derived from the matrix representation X of x. But our approach underlines the correspondence between $\mathscr{C}_2^{(n)}$ and $SL(n, \mathbb{C})$ and allows the extension to $\mathscr{C}_2^{(n)}$ of our results concerning the polar representation of GCA (Fleury *et al.*, 1991). In this connection, a systematic study of the exponential representation of the $\mathscr{C}_1^{(n)}$ algebra has been entered upon recently by Kwasniewski (1992).

The connection between GCA and $SU(n)$ is not new and was already mentioned by Ramakrishnan *et al.* (1969).

The quaternions allow a description of the composition of rotations in $\mathbb{R}^3$. In the same way, the generalized quaternions offer a description of the composition law of $SL(n, \mathbb{C})$ elements. But there is a major difference between the two cases. In the former case, quaternions and $SU(2)$ have the same generators (in other words, the algebra of Pauli matrices is closed under commutation and anticommutation relations) and this is no longer true in the latter case as soon as $n>2$.

To conclude the section, let us mention that for an element of $\mathscr{C}_2^{(m|n)} \otimes \mathscr{C}_2^{(m-n|n)}$ the invariance group reduces to

$$SL_L(m, \mathbb{C}) \otimes SL_L(n-m, \mathbb{C}) \otimes SL_R(m, \mathbb{C}) \otimes SL_R(n-m, \mathbb{C})$$

and in the special case (2.5) simply to $SL_L(2, \mathbb{C}) \otimes SL_R(2, \mathbb{C})$.

3. CONNECTION TO HYPERSPIN MANIFOLDS

Up to now, the element $x = \sum_{a,b=0}^{n-1} x_{ab} e_1^a e_2^b$ has not been subjected to any condition. Consider the question of Hermicity and define the Hermitian conjugation

$$(e_1^a e_2^b)^+ = e_2^{-b} e_1^{-a} \tag{3.1}$$

The set of Hermitian generalized Clifford numbers define an n^2 vectorial space over $\mathbb{R}$:

$$\mathscr{C}_2^{(n)\text{Herm}} = \{x \in \mathscr{C}_2^{(n)};\ x^+ = x\} \tag{3.2}$$

the Hermitian basis of which is

$$R_{ab} = \tfrac{1}{2}(e_1^a e_2^b + e_2^{-b} e_1^{-a}) \tag{3.2a}$$

$$a, b = 0, \ldots, n-1$$

$$I_{ab} = -\frac{i}{2}(e_1^a e_2^b - e_2^{-b} e_1^{-a}) \tag{3.2b}$$

It is no longer an $\mathbb{R}$-algebra. A similar problem was solved by Jordan (1932), who introduced a modified multiplication law:

$$x \cdot y = \tfrac{1}{2}(xy + yx)$$

This commutative product is not associative:

$$(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z) \neq 0$$

But it is flexible,

$$(x, y, x) = 0$$

and fulfills the Jordan identity,

$$(x^2, y, x) = 0$$

A representation of the Jordan algebra can thus be obtained from a GCA.

For $x^+ = x$, the transformation law has to preserve the Hermicity of x. The consequence is that $R_L = R_R^+$ and thus the symmetry group reduces to $SL(n, \mathbb{C})$. This was the starting point of Finkelstein (1986) and Finkelstein *et al.* (1986) when they introduced the concept of hyperspin manifold, the aim of which was to reconcile two different theories:

(a) The spin manifold which allows a spinorial decomposition of space-time (see, for instance, Penrose and Rindler, 1984). In such a description, one gets a $2^n D$ space-time.
(b) The Kaluza–Klein theory, which leads, after compactification, to a $4D$ manifold describing gravitation as well as Yang–Mills interactions (see, for instance, Duff *et al.*, 1986, and references therein). In such a description we have an arbitrary-dimensional space and realistic Kaluza–Klein theories are not constructed using a 2^n-dimensional space (Witten, 1981).

With the spinorial construction of the Minkowski space-time as a guidance, these authors have considered a complex n-dimensional space Σ_n, namely the hyperspinor space, and hyperspinors belong to the $\underset{\sim}{n}$ representation of $SL(n, \mathbb{C})$. With the use of the inequivalent complex conjugate representation Σ_n^+, they defined the n^2 hyperspin manifold $\Sigma_n \otimes \Sigma_n^+$. For an element x of this space, they then built an n-linear metric η:

$$\|x\|^n = \eta_{i_1 \ldots i_n} x^{i_1} \ldots x^{i_n}, \qquad 1 \leq i, j \leq n^2$$

by an appropriate determinant invariant under $SL(n, \mathbb{C})$. Since Σ_2 (the usual spinorial space) is naturally included in Σ_n, they obtained the embedding of the Minkowski space-time in the hyperspin manifold and meanwhile of the Lorentz group in $SL(n, \mathbb{C})$.

With the results established in Section 2, we propose an algebraic interpretation of the hyperspin approach in terms of GCA through the identification

$$\Sigma_n \otimes \Sigma_n^+ = \mathscr{C}_2^{(n)\mathrm{Herm}} \tag{3.3}$$

Taking $m = 2$ and $\lambda = 1$ in (1.7), we obtain the embedding described above, which can be understood by the inclusions of Pauli numbers into the Hermitian generalized quaternions. Using the transformation property (1.2), one can rewrite $\Delta(x)$ in the form they proposed:

$$\Delta(x)=\begin{pmatrix} x_0+x_3 & x_{12}+iy_{12} & \cdots & x_{1n}+iy_{1n} \\ x_{12}-iy_{12} & x_0-x_3 & & \vdots \\ \vdots & & \lambda_3 & \vdots \\ x_{1n}-iy_{1n} & & & \lambda_n \end{pmatrix}$$

To conclude, we make the identification between the hyperspinors and the extended spinors of Popovici and Ghéorghe (1966*b*). Those extended spinors had been introduced by considering GCA in a way similar to the construction of spinors from the Clifford algebra by Brauer and Weyl (1935).

4. CONCLUSION

The GCA $\mathscr{C}_2^{(n)}$ has been provided with an $n-ic$ metric invariant under $SL(n, \mathbb{C})\otimes SL(n, \mathbb{C})$. In the Hermitian limit, the symmetry group reduces to $SL(n, \mathbb{C})$, and for $\mathscr{C}_2^{(m)}\otimes\mathscr{C}_2^{(n-m)}$ to $SL(m, \mathbb{C})\otimes SL(n-m, \mathbb{C})$. For the peculiar case $m=2$ and with a unit element in $\mathscr{C}_2^{(n-m)}$ together with the Hermitian limit, one gets the Minkowski space-time as an inclusion of the hyperspin manifold. The same procedure applied to GCA $\mathscr{C}_2^{(p)}$ with p canonical generators leads to an $n-ic$ metric on an n^p-dimensional manifold. This seems to indicate some kind of analogy between the $\mathscr{C}_p^{(2)}$ and $\mathscr{C}_p^{(n)}$ algebras which is strengthened by the matrix representation of $\mathscr{C}_p^{(n)}$ (Moringa and Nono, 1952; Morris, 1967; Fleury and Rausch de Traubenberg, 1992) analogous to the one used by Brauer and Weyl (1935) to obtain spinors in an arbitrary-dimensional space-time. However, this analogy has its limitations:

(i) The Pauli matrices linearize $x^2+y^2+z^2$, polynomial which is $SO(3)$ invariant, and $SO(3)$ is included in $SO(4)$, the invariance group of quaternions. But the generalized Pauli matrices allow the linearization of $x^n+y^n+z^n$, which is a polynomial having a discrete symmetry group.

(ii) Pauli matrices are sufficient to linearize any quadratic form, but extended Pauli matrices cannot linearize any $n>2$ form. Other matrices have to be taken into consideration (Fleury and Rausch de Traubenberg, 1992).

(iii) When one calculates

$$\exp \sum_{\substack{a,b=0\\(a,b)\neq(0,0)}}^{1} {}_i\varphi_{ab}\sigma_1^a\sigma_3^b \equiv \exp {}_i\boldsymbol{\varphi}\cdot\boldsymbol{\sigma}$$

one gets a closed formula:

$$\exp(i\boldsymbol{\varphi}\cdot\boldsymbol{\sigma})=\cos(\boldsymbol{\varphi}\cdot\boldsymbol{\varphi})+\sin(\boldsymbol{\varphi}\cdot\boldsymbol{\varphi})\frac{i\boldsymbol{\varphi}\cdot\boldsymbol{\sigma}}{\boldsymbol{\varphi}\cdot\boldsymbol{\varphi}}$$

This is not true for $\mathscr{C}_2^{(n)}$ $(n>2)$.

(iv) For $n=2$, one has a coincidence between the generators of $SU(2)$ and the three imaginary units of the quaternions. This is no longer true for $n>2$.

To summarize, one can say that the properties lost when passing from quaternions to n^2-ions are analogous to those lost when passing from quadratic forms to n-forms. This is the same thing *mutatis mutandis* for our $\mathscr{C}_2^{(2)}$ and $\mathscr{C}_2^{(n)}$ GCA.

ACKNOWLEDGMENTS

We thank C. Schubert for useful remarks and discussions. One of us (M.R.T.) thanks the Joint Institute for Nuclear Research, Dubna, for its hospitality.

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